International Journal of advancement in electronics and computer engineering (IJAECE)
Volume 3, Issue 4, July 2014, pp.313-316, ***ISSN 2278 -1412***
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A Fixed Point Theorem for Multivalued Mapping in Uniform Space

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Abstract – *The aim of this paper to prove a fixed point theorem for multivalued mapping in orbitally complete Hausdorff uniform space and hyperspace. These results generalize several corresponding relations in uniform space.*

Keywords: *Fixed point, uniform space, multivalued mapping, orbitally complete,*

I. Introduction

The well known Banach fixed point theorem for contraction mapping has been generalized and extended in many direction. Acharya[1], Mishra and Singh[2], Turkoglu and Fisher[3],have given many interesting results of fixed point theory in uniform space. Our results generalize the corresponding result of Badshah and Sharma[4].Since the uniform space form a natural extension of the matric space.there exists considerable literature of fixed point theory dealing with results on fixed point or common fixed points in uniform spaces(see for ex.[5]-[9]).

II. Definition

II.1.Definition 1.1[4]

A uniformity for a set X is a non-void family U of substs of $X \times X$ such that:

(1.1.1) each member of U contains the diagonal Δ,
(1.1.2) if $u \in U$ then $u^{-1} \in U$,
(1.1.3) if $u \in U$ then $v \circ v \subseteq U$, for some $v \in U$
(1.1.4) if u and v are members of U , then $u \cap v \in U$ and
(1.1.5) if $u \in U$ and $u \subseteq v \subseteq X \times X$, then $v \in U$.

The pair (X, U) is called a uniform space.

II.2. Definition.1.2 [3]

Let (X, U) be a uniform space. family $\{d_i : i \in I\}$ of pseudometric on X with indexingset I , is called an associated family for the uniformity U if the family

$$\beta = \{V(i, \varepsilon) : i \in I; \varepsilon > 0\},$$

Where,

$$V(i, \varepsilon) = \{(x, y) : x, y \in X, d_i(x, y) < \varepsilon\}$$

is the sub-base for the uniformity U .We may assume that β itself is a base by adjoining finite intersection of members of β ,if necessary. The corresponding family of pseudometric is called an associated family for U .An associated family for U will be denoted by p^*.

II.3.Definition1.3 [4]

Let A be a nonempty subset of a uniform space X .Define

$$\Delta^*(A) = \sup\{d_i(x, y) : x, y \in A, i \in I\},$$

Where

$$\{d_i : i \in I\} = p^*,$$

Then Δ^* is called an augmented diameter of A . Further, A is said to be p^* - bounded if $\Delta^*(A) < \infty$.

IV.4.Definition.1.4 [3]

Let 2^X = {A: A is a nonempty, Closed and p^*-bounded subset of X}. For any nonempty subsets A and B of X, define

$$d_i(X,A)=\inf\{d_i(x,a): a\in A\}, i\in I$$

$$H_i(A,B)=\max\left\{\sup_{a\in A} d_i(a,B), \sup_{b\in B} d_i(A,b)\right\}$$

$$=\sup_{x\in X}\{|d_i(x,A)-d_i(x,B)|\}.$$

It is well known that on 2^X, H_i is a pseudometric, called the Hausdorff pseudometric induced by d_i, $i\in I$.

V.5.Definition.1.5 [3]

Let (X,U) be a uniform space with an augmented associated family p^*. p^* also induces a uniformity U^* on 2^X defined by the base

$$\beta^*=\{V^*(i,\varepsilon): i\in I, \varepsilon>0\},$$

Where

$$V^*(i,\varepsilon)=\{(A,B): A,B\in 2^X, H_i(A,B)<\varepsilon\}.$$

The space $(2^X,U^*)$ is a uniform space called the hyperspace of (X,U).

VI.6.Definition.1.6 [3]

The collection of all filters on a given set X is denoted by $\phi(X)$. An order relation is define on $\phi(X)$ by the rule $F_1<F_2$ if $F_1\supset F_2$. If $F^*<F$, then F^* is called a sub-filter of F.

VII.7.Definition.1.7 [3]

Let (X,U) be a uniform space defined by $\{d_i: i\in I\}=p^*$. If $F: X\to 2^X$ is a multivalued mapping, then

(i) $x\in X$ is called a fixed point of F if $x\in Fx$.

(ii) An orbit of F at a point $x_0\in X$ is a sequence $\{x_n\}$ given by

$$O(F,x_0)=\{x_n : x_n\in Fx_{n-1}, n=1,2,...\};$$

(iii) A uniform space X is called F-orbitally complete if every Cauchy filter which is a subfilter of an orbit of F at each $x\in X$ converges to a point of X.

VIII.8.Definition.1.8 [3]

Let (X,U) be a uniform space and let $F: X\to X$ be a mapping. A single valued mapping F is orbitally continuous if $\lim(T^{n_i}x)=u$ implies $T(T^{n_i}x)=Tu$ for each $x\in X$.

III. Main Results

III.1.Theorem.1.1

Let (X,U) be an F-orbitally complete Hausdorff uniform space defined by $\{d_i: i\in I\}=p^*$ and $(2^X, U^*)$ a hyperspace and let $F: X\to 2^X$ be a continuous mapping with Fx compact for each $x\in X$ Assume that

$$\min\{H_i(Fx,Fy)^r, d_i(x,Fx)d_i(y,Fy)^{r-1}, d_i(y,Fy)^r\}+ a_i\min\left\{\frac{d_i(x,Fy)d_i(y,Fx)}{d_i(x,y)}, \frac{d_i(Fx,Fy)d_i(y,Fx)}{d_i(x,y)}\right\}$$

$$\le\left[b_i d_i(x,Fx)+c_i d_i(x,y)+e_i d_i(y,Fx)\left\{\frac{1+d_i(x,Fy)}{1+d_i(x,y)}\right\}\right]d_i(y,Fy)^{r-1} \quad \ldots(1)$$

For all $i\in I$ and $x,y\in X$, where $r\ge 1$ is an integer a_i, b_i, c_i, e_i are real numbers such that $0<b_i+c_i<1$ then F has a fixed point.

Proof: Let x_0 be an arbitrary point in X and consider the sequence $\{x_n\}$ defined by

$$x_1\in Fx_0, x_2\in Fx_1..., x_n\in Fx_{n-1},...$$

Let us suppose that $d_i(x_n,Fx_n)>0$ for each $i\in I$ and $n=0,1,2,...$ Let $U'\in U$ be an arbitrary entourage. Since β is a base for U, there exists $V(i,\varepsilon)\in\beta$ such that $V(i,\varepsilon)\subseteq U'$. Now $y\to d_i(x_0,y)$ is continuous on the compact set Fx_0 and this implies that there exists $x_1\in Fx_0$ such that $d_i(x_0,x_1)=d_i(x_0,Fx_0)$. Similarly, Fx_1 is compact so there exists $x_2\in Fx_1$ such that $d_i(x_1,x_2)=d_i(x_1,Fx_1)$. Continuing, we obtain a sequence $\{x_n\}$ such that $x_{n+1}\in Fx_n$ and $d_i(x_n,x_{n+1})=d_i(x_n,Fx_n)$.

For $x = x_{n-1}$, and $y = x_n$ by the condition (1), we have

$$\min\{H_i(Fx_{n-1}, Fx_n)^r, d_i(x_{n-1}, Fx_{n-1})d_i(x_n, Fx_{n-1})^{r-1}, d_i(x_n, Fx_n)^r\}$$

$$+ a_i \min\left\{\frac{d_i(x_{n-1}, Fx_n)d_i(x_n, Fx_{n-1})}{d_i(x_{n-1}, x_n)}, \frac{d_i(Fx_{n-1}, Fx_n)d_i(x_n, Fx_{n-1})}{d_i(x_{n-1}, x_n)}\right\}$$

$$\leq \left[b_i d_i(x_{n-1}, Fx_{n-1}) + c_i d_i(x_{n-1}, x_n) + e_i d_i(x_n, Fx_{n-1})\left\{\frac{1+d_i(x_{n-1}, Fx_n)}{1+d_i(x_{n-1}, x_n)}\right\}\right] d_i(x_n, Fx_n)^{r-1}$$

Since $d_i(x_n, Fx_{n-1}) = 0, x_n \in Fx_{n-1}$, hence we have

$$\min\{d_i(x_n, x_{n+1})^r, d_i(x_{n-1}, x_n)d_i(x_n, x_{n+1})^{r-1}\} \leq$$
$$[b_i d_i(x_{n-1}, x_n) + c_i d_i(x_{n-1}, x_n)]d_i(x_n, x_{n+1})^{r-1}$$

It follows that-:

$$\min\{d_i(x_n, x_{n+1})^r, d_i(x_{n-1}, x_n)d_i(x_n, x_{n+1})^{r-1}\} \leq$$
$$[(b_i + c_i)d_i(x_{n-1}, x_n)]d_i(x_n, x_{n+1})^{r-1}$$

Since

$$d_i(x_{n-1}, x_n)d_i(x_n, x_{n+1})^{r-1} \leq$$

$[(b_i + c_i)d_i(x_{n-1}, x_n)]d_i(x_n, x_{n+1})^{r-1}$ is not possible (as $0 < b_i + c_i < 1$), we have

$$d_i(x_n, x_{n+1})^r \leq [b_i + c_i)d_i(x_{n-1}, x_n)]d_i(x_n, x_{n+1})^{r-1}$$
$$d_i(x_n, x_{n+1})^r \leq k_i d_i(x_{n-1}, x_n)d_i(x_n, x_{n+1})$$

Where,

$$k = b_i + c_i \qquad 0 < k_i < 1$$

Proceeding in this manner, we get

$$d_i(x_n, x_{n+1}) \leq k_i d_i(x_{n-1}, x_n)$$
$$\leq k_i^2 d_i(x_{n-2}, x_{n-1}) \leq \ldots \leq k_i^n d_i(x_0, x_1).$$

Hence, we obtain

$$d_i(x_n, x_m) \leq d_i(x_n, x_{n+1}) + d_i(x_{n+1}, x_{n+2}) + \ldots + d_i(x_{m-1}, x_m)$$
$$\leq (k_i^n + k_i^{n+1} + \ldots + k_i^{m-1})d_i(x_0, x_1)$$

$$\leq k_i^n(1 + k_i + \ldots + k_i^{m-n-1})d_i(x_0, x_1)$$
$$\leq \frac{k_i^n}{1-k} d_i(x_0, x_1).$$

Since $\lim_{n\to\infty} k_i^n = 0$, it follows that there exists $N(i, \varepsilon)$ such that $d_i(x_n, x_m) < \varepsilon$ and hence $(x_n, x_m) \in U'$ for all $n, m \geq N(i, \varepsilon)$. Therefore the sequence $\{x_n\}$ is a Cauchy sequence in the d_i -uniformity on *X*.

Let $S_p = \{x_n : n \geq p\}$ for all positive integers p and let β be the filter basis $\{S_p : p = 1,2,\ldots\}$. Then since $\{x_n\}$ is a d_i -Cauchy sequence for each $i \in I$, it is easy to see that the filter basis β is a Cauchy filter in the uniform space (X, U). To see this, we first note that the family $\{V(i, \varepsilon) : i \in I\}$ is a base for U as $p^* = \{d_i : i \in I\}$. Now since $\{x_n\}$ is a d_i -Cauchy sequence in *X*, there exists a positive integer p such that $d_i(x_n, x_m) < \varepsilon$ for $m \geq p, n \geq p$. This implies that $S_p \times S_p \subseteq V(i, \varepsilon)$. Thus, given any $U' \in U$, we can find an $S_p \in \beta$ such that $S_p \times S_p \subset U'$. Hence β is Cauchy filter in (X, U). Since (X, U) is F-orbitally complete and Hausdorff space, $S_p \to z$ for some $z \in X$. Consequently, $F(S_p) \to Fz$. Also

$$S_{p+1} \subset F(S_p) = \cup\{Fx_n : n \geq p\}$$

For $p = 1, 2, \ldots$ it follows that $z \in Fz$. Hence z is a fixed point of F. This completes the proof.

If we take $r = 1$ in Theorem .1, then we obtain the following theorem.

III.2.Theorem1.2

Let (X, U) be an *F*-orbitally complete Hausdorff

$$a_i \min\left\{\frac{d_i(z, Tw)d_i(w, Tz)}{d_i(z, w)}, \frac{d_i(Tz, Tw)d_i(w, Tz)}{d_i(z, w)}\right\}$$

uniform space defined by $\{d_i : i \in I\} = p^*$ and $(2^X, U^*)$

a hyperspace and let $F : X \to 2^X$ be a continuous mapping with Fx compact for each $x \in X$. Assume that

$$\min\{H_i(Fx, Fy), d_i(x, Fx), d_i(y, Fy)\} +$$

$$a_i \min\left\{\frac{d_i(x, Fy)d_i(y, Fx)}{d_i(x, y)}, \frac{d_i(Fx, Fy)d_i(y, Fx)}{d_i(x, y)}\right\}$$

$$\leq \left[b_i d_i(x, Fx) + c_i d_i(x, y) + e_i d_i(y, Fx)\left\{\frac{1+d_i(x, Fy)}{1+d_i(x, y)}\right\}\right] .(2)$$

For all $i \in I$ and $x, y \in X$, where a_i, b_i, c_i, e_i are real numbers such that $0 < b_i + c_i < 1$, then F has a fixed point.

We denote that if *F* is a single valued mapping on *X*, then we can write

$$d_i(Fx, Fy) = H_i(Fx, Fy), x, y \in X, i \in I.$$

Thus we obtain the following theorem as a consequence of the Theorem (2).

III.3.3Theorem 1.3

Let (X,U) be a T -orbitally complete Hausdorff uniform space let $T:X\to X$ be a T orbitally continuous mapping satisfying

$$\min\{d_i(Tx,Ty),d_i(x,Tx),d_i(y,Ty)\}+$$

$$a_i\min\left\{\frac{d_i(x,Ty)d_i(y,Tx)}{d_i(x,y)},\frac{d_i(Tx,Ty)d_i(y,Ty)}{d_i(x,y)}\right\}$$

$$\leq\left[b_id_i(x,Tx)+c_id_i(x,y)+e_id_i(y,Tx)\left\{\frac{1+d_i(x,Ty)}{1+d_i(x,y)}\right\}\right] \quad \ldots(3)$$

for all $i\in I$ and $x,y\in X$, where a_i,b_i,c_i,e_i are real numbers such that $0<b_i+c_i<1$, then T has a fixed point and which is unique whenever $a_i>c_i+e_i>0$

Proof-: Define a mapping F of X into 2^X by putting $F_x=\{T_x\}$ for all x in$X$.It follows that F satisfies the condition of Th.2. Hence T has a fixed point.
Now if $a_i>c_i+e_i>0$, we show that T has a unique fixed point.Assume that T has two fixed point z and w which are distinct.Since $d_i(z,T_z)=0$ and $d_i(w,T_w)=0$, then by the condition (3),

$$a_i\min\left\{\frac{d_i(z,Tw)d_i(w,Tz)}{d_i(z,w)},\frac{d_i(Tz,Tw)d_i(w,Tz)}{d_i(z,w)}\right\}$$

$$\leq\left[b_id_i(z,Tz)+c_id_i(z,w)+e_id_i(w,Tz)\left\{\frac{1+d_i(x,Tw)}{1+d_i(x,w)}\right\}\right]$$

$$a_i\min\left\{\frac{d_i(z,w)d_i(w,z)}{d_i(z,w)},\frac{d_i(z,w)d_i(w,z)}{d_i(z,w)}\right\}$$

$$\leq\left[b_id_i(z,z)+c_id_i(z,w)+e_id_i(w,z)\left\{\frac{1+d_i(x,w)}{1+d_i(x,w)}\right\}\right]$$

$$a_i\min\{d_i(z,w),d_i(w,z)\}\leq(c_i+e_i)d_i(z,w)$$

$$a_id_i(z,w)\leq(c_i+e_i)d_i(z,w)$$

$$d_i(z,w)\leq\frac{(c_i+e_i)}{a_i}d_i(z,w)$$

This is impossible. Thus if $a_i>c_i+e_i>0$. Then T has a unique fixed point in X. This completes the proof.

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